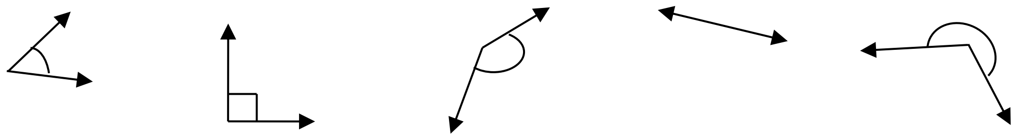
Module 8: Trigonometry

SOME BASIC GEOMETRY

Types of angles:

Exercise 1: Match each angle type with the correct definition.

Acute **Right Angle** **Obtuse** **Straight line** **Reflex**



a. Acute Angle
b. Right Angle
c. Obtuse Angle
d. Straight Line
e. Reflex Angle

I. Exactly 90°
II. Less than 90°
III. Exactly 180°
IV. More than 180°
V. Between 90° and 180°

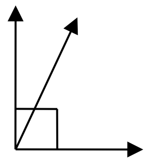
Angle Relationships:

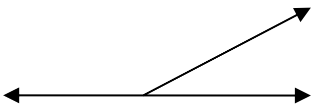
Complementary Angles

Add to 90°

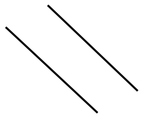
Supplementary Angles

Add to 180°

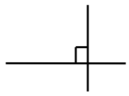




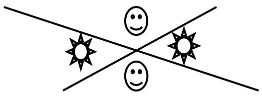
Parallel lines do not intersect.



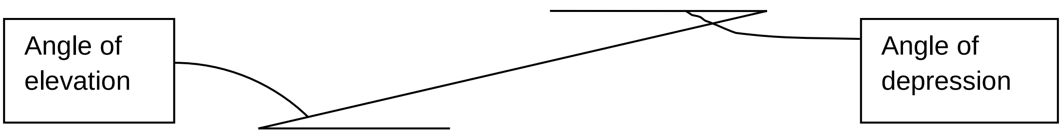
Perpendicular lines intersect at right angles.



Opposite angles are formed by intersecting lines and have the same angles (i.e. are congruent).



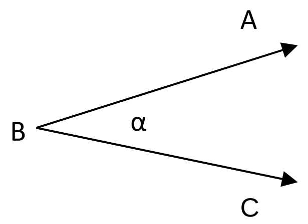
An **angle of elevation** is measured up from the horizontal.



NAMING ANGLES

An angle is formed by two lines intersecting at a common point. The common point is called the vertex.
Angles can be named in four different ways:

<B, α,<ABC or <CBA. When three letters are used to name an angle, the vertex is the middle letter.

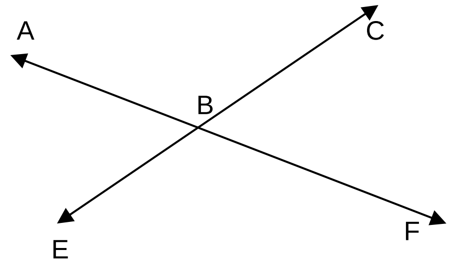


Exercise 2

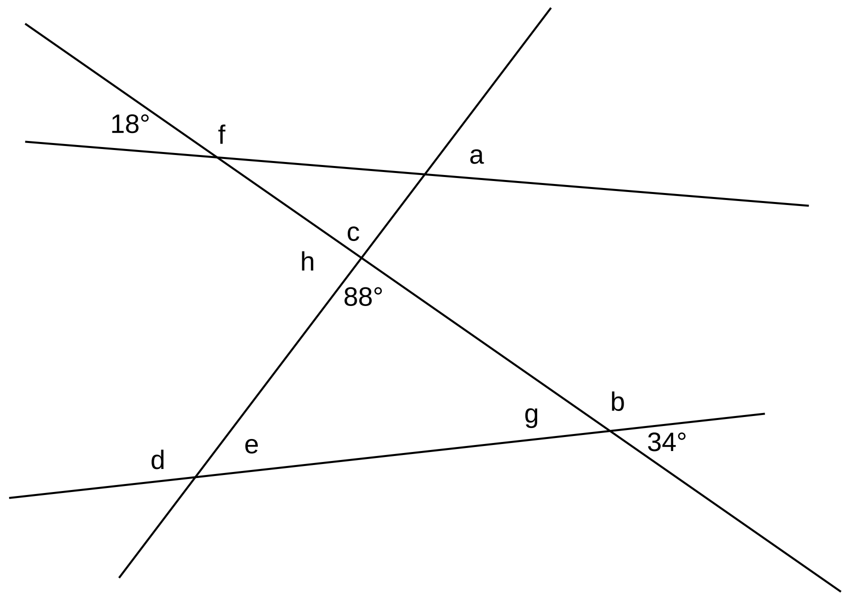
1. Using three letters, name two obtuse angles and two acute angles in the diagram below.

Two obtuse angles are: ___________________

Two acute angles are: ___________________



1. Use your knowledge of angle relationships and the fact **angles of a triangle sum to 180°** to find the value of every angle in the diagram below.



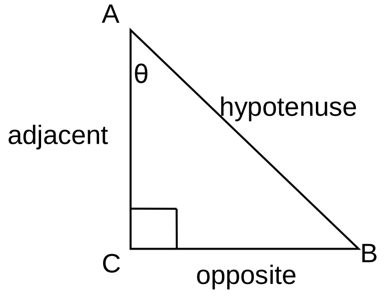
a=_____________ b=_____________ c=_____________ d=______________

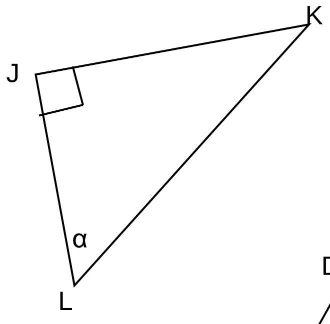
e=_____________ f=_____________ g=_____________ h=______________

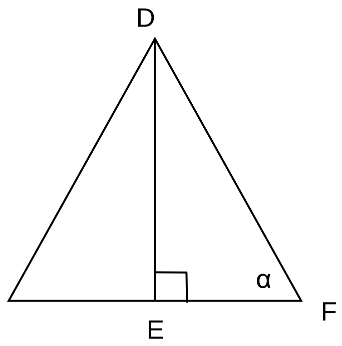
NAMING THE SIDES OF A RIGHT TRIANGLE

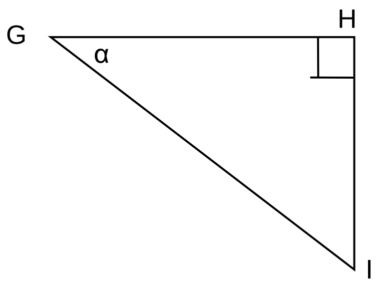
Any right triangle can be named with reference to one of the acute angles. In the **example** the sides are named opposite and adjacent with reference to the angle θ. The side opposite the right angle is always referred to as the hypotenuse.

Worked example









Exercise 3

On each of the three right angle triangles ΔJKL, ΔGHI and ΔEDF;

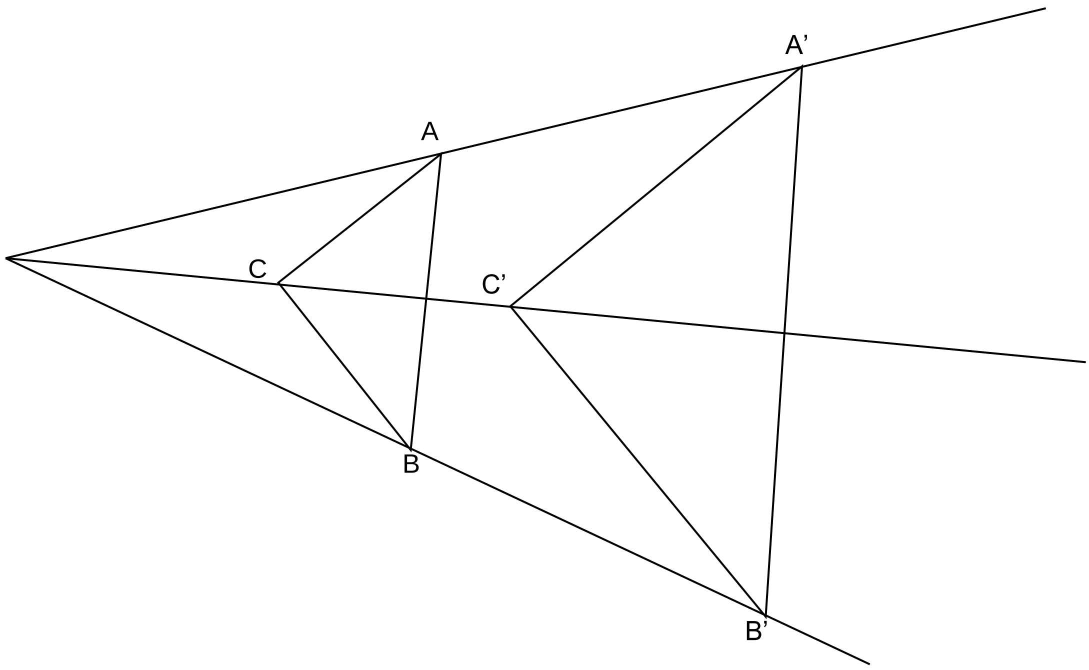
a. Label the hypotenuse.

b. Label the side which is opposite to the angle α (alpha).

c. Label the side which is adjacent to the angle α.

SIMILAR TRIANGLES

In geometry, two figures are similar if their corresponding angles are congruent (i.e. the same) and the lengths of the corresponding sides are in the same ratio. Consider the following two examples which involve similar triangles.



Exercise 4

1. Imagine a light source is used to project an image of a triangle onto the wall at the front of a seminar room, and we are asked to calculate the scale factor of the projected image.

a. The lengths of AB, BC and CA are 5, 4 and 3 respectively. The lengths of A'B', B'C' and C'A' are 10, 8 and 6 respectively. Calculate the ratios of the corresponding sides and hence the scale factor for the image.

$\frac{A'B'}{AB} =$ $\qquad$ $\frac{B'C'}{BC} =$ $\qquad$ $\frac{C'A'}{CA} =$ $\qquad$ $\therefore$ Scale Factor=

b. If the light source is moved so that the lengths of A'B', B'C' and C'A' increase to 17.5, 14 and 10.5 respectively. Calculate the new the scale factor for the image.

$\frac{A'B'}{AB} =$ $\qquad$ $\frac{B'C'}{BC} =$ $\qquad$ $\frac{C'A'}{CA} =$ $\qquad$ $\therefore$Scale Factor=

2. A tree casts a shadow of 23 metres along horizontal ground; a nearby fence post 2 metres high casts a shadow 5 metres long. Find the height of the tree.

a. Draw a picture of the problem and identify the similar sides.

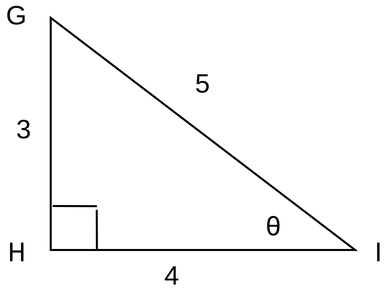
b. The corresponding sides are proportional (i.e. have the same ratio). Write down the two ratios that must be equal to each other. Solve the equation for the unknown height of the tree.

TRIGONOMETRIC RATIOS

Sine ratio= $\frac{\textbf{opposite}}{\textbf{hypothenuse}}$

Cosine ratio= $\frac{\textbf{adjacent}}{\textbf{hypothenuse}}$

Tangent Ratio = $\frac{\textbf{opposite}}{\textbf{adjacent}}$

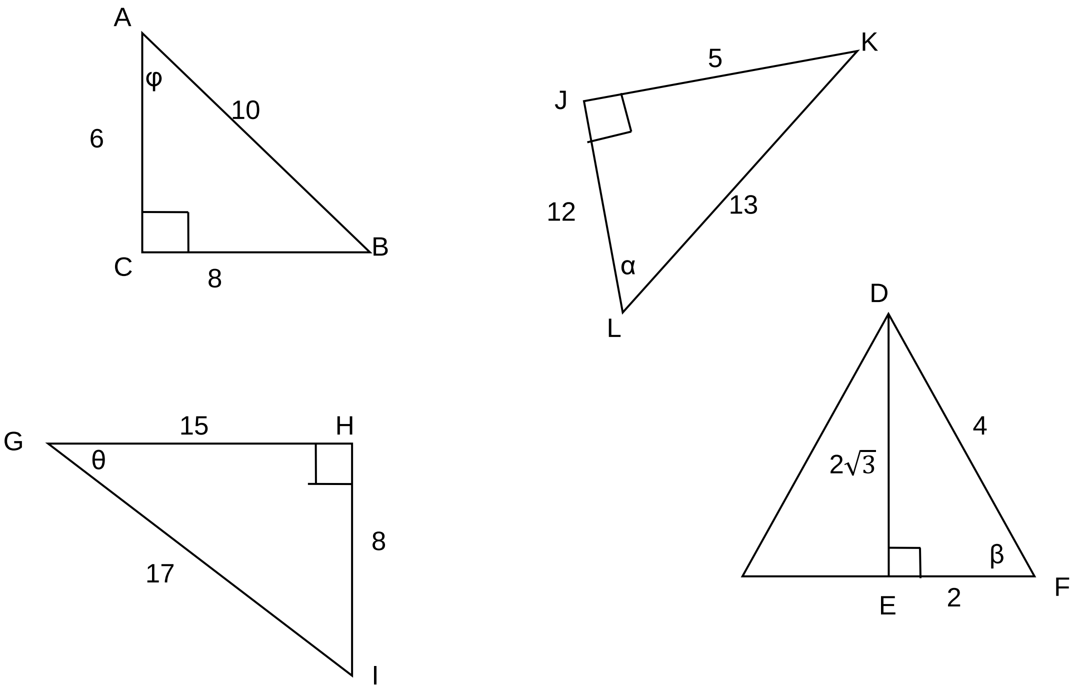


For this right triangle, $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$ and $\tan\theta = \frac{3}{4}$.

The property of right angle triangles, that the ratios of the sides remain the same for a given angle, whatever the scale of the triangle that the angle is in, makes the trigonometric ratios useful in many practical situations.

Exercise 5

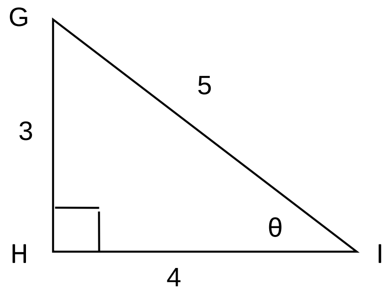
In the table provided, write down the trigonometric **ratios**, for the labelled angles, in each of the following right triangles. (Calculator not required.)



Trig. Ratios	**φ**	**α**	**θ**	**β**
Sine	$\frac{8}{10} = \frac{4}{5}$			
Cosine		$\frac{12}{13}$		
Tangent				$\frac{2\sqrt{3}}{2} = \sqrt{3}$

In the next two sections, we will learn how we can use the trig ratios to find the unknown length of the side of a triangle or the size of an angle in a triangle.

FINDING AN UNKNOWN ANGLE OF A RIGHT ANGLE TRIANGLE



Sine ratio= $\frac{\textbf{opposite}}{\textbf{hypothenuse}}$

For this right triangle, we write sin θ = $\frac{3}{5}$
Use the inverse sine button $\sin^{-1}$
on your calculator to find θ.
We write θ = $\sin^{-1} \frac{3}{5}$

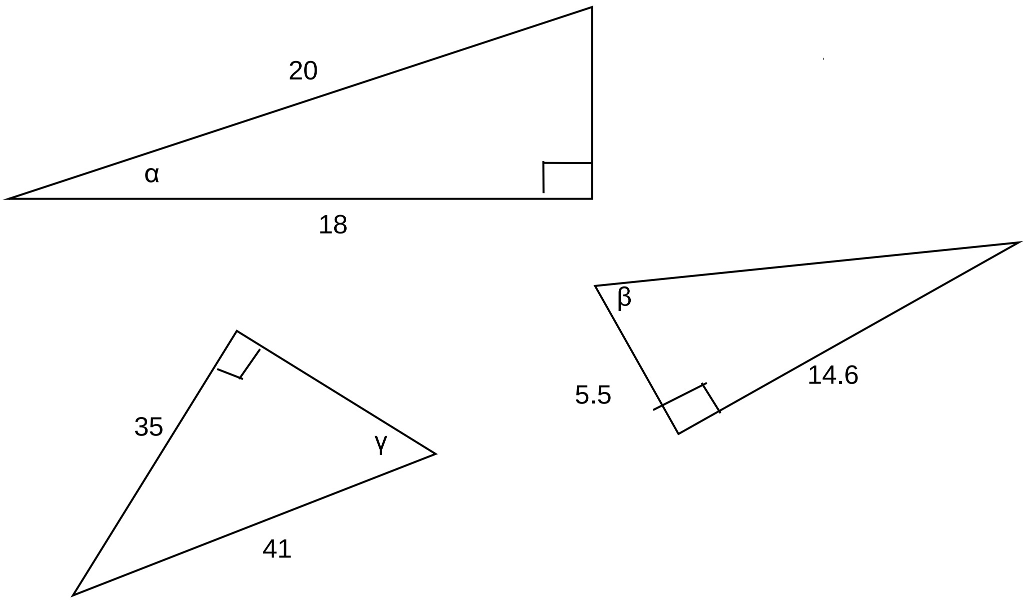
First, check your calculator is set to degrees, now convert $\frac{3}{5}$ to a decimal 3 ÷ 5 = 0.6. Find the $\sin^{-1}$ button on your calculator $\sin^{-1} 0.60$ = 36.87° (Answer rounded to 2 d.p.). Now find $\sin^{-1}(\frac{3}{5})$, do not convert $\frac{3}{5}$ to a decimal, you should get the same answer.

Cosine ratio= $\frac{\textbf{adjacent}}{\textbf{hypothenuse}}$

For the same right triangle, we write cos θ = $\frac{4}{5}$. Use the inverse cosine button $\cos^{-1}$ on your calculator to find θ. We write θ = $\cos^{-1} \frac{4}{5}$. Find the $\cos^{-1}$ button on your calculator, $\cos^{-1}(\frac{4}{5})$ = 36.87°.

Tangent Ratio = $\frac{\textbf{opposite}}{\textbf{adjacent}}$

Again, using the same right triangle, we write tan θ = $\frac{3}{4}$. Use the inverse tangent button $\tan^{-1}$ on your calculator to find θ. We write θ = $\tan^{-1} \frac{3}{4}$. Find the $\tan^{-1}$ button on your calculator, $\tan^{-1}(\frac{3}{4})$ = 36.87°.



Worked example

In the first triangle above, the unknown angle α can be found by using the cosine ratio. This is because the length of the side adjacent to α is 18 and the length of the hypotenuse is 20.

$\cos \alpha = 18/20$

$\cos \alpha = 0.9$

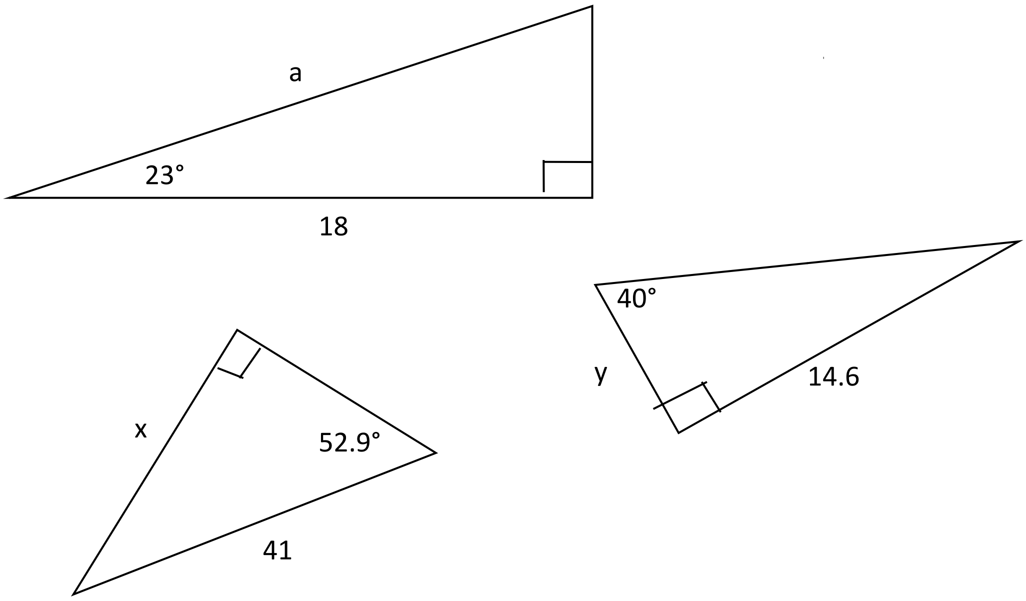
$\alpha = \cos^{-1} 0.9 = 25.8^{\circ}$ (Make sure your calculator is set to give the answer in degrees, if your answer is .45, your calculator is set to give the answer in radians)

Exercise 6

1. Find the unknown angles ''β'' and ''γ'' in the other two triangles above.

2. In Exercise 3, question 2 above, the length of the shadow cast by a tree was 23 metres, we calculated the height of the tree to be 9.2 metres. Draw a new diagram and calculate the angle of elevation of the sun at this time of day?

FINDING AN UNKNOWN SIDE OF A RIGHT ANGLE TRIANGLE



Worked example

In the first triangle, the unknown side ‘’a’’, can be found by using the cosine ratio. This is because the cos 23°=18/a.

Use your calculator to confirm that cos 23°=.9205. Check your calculator set to ‘degrees’.

Rearranging the equation we get a=18/cos 23°.

Therefore a = 18/cos 23° = 19.6 units.

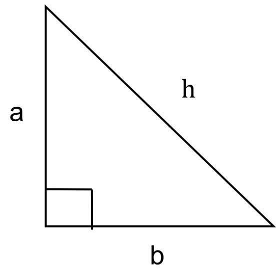
Exercise 7

1. Find the unknown length of the sides ‘’x’’ and ‘’y’’ in the other two triangles above.

PYTHAGORAS' THEOREM

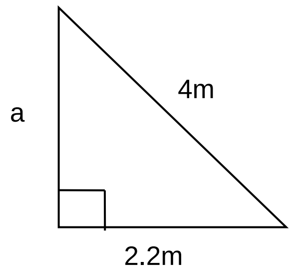
Pythagoras' theorem can be used to find the unknown length of a side of a right angle triangle, when the length of the other two sides are known.

For a right angle triangle with sides of length a and b, and hypotenuse of length h, then; $h^2 = a^2+b^2$



Worked example

A ladder is resting against a wall. It is 4metres in length is 2.2metres from the base of the wall. Find how far the top of the ladder is from the ground, to the nearest centimetre.



$4^2 = a^2 + 2.2^2$

$a^2 = 4^2 - 2.2^2$

$a^2 = 16 - 4.84$

$a^2 = 11.16$

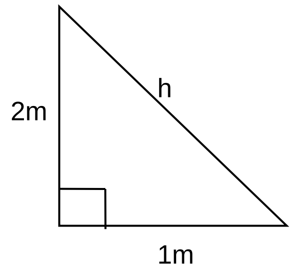
$a = \sqrt{11.16}$

$a = 3.34m = 334cm$

Exercise 8

1. A triangle has sides of length 5, 12 and 13, is it a right triangle?

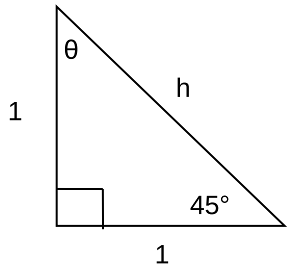
2. A rectangular wooden gate is 2 metres high and 1 metre wide, with a diagonal piece of wood to strengthen the gate. Find the length of the diagonal, to the nearest centimetre.

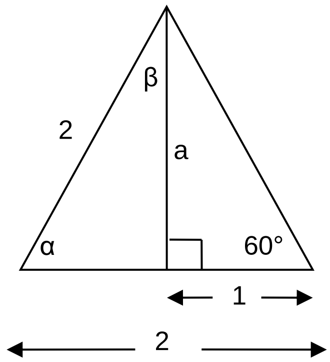


3. A 24m long rectangular swimming pool has a diagonal length of 25m. What is the width of the swimming pool?

Useful ratios – the sin, cos and tan of 30°. 45° and 60°

4. The 90°, 45°, 45° triangle pictured below on the left has two sides of length 1 unit. From this triangle, the sin, cos and tan of 45° can be found.
   The equilateral (60°, 60°, 60°) triangle pictured on the right has sides of length 2. If we divide this triangle in half, the base has a length of 1 and we have two 90°, 60°, 30° triangles. From the 90°, 60°, 30° triangle we can find the sin, cos and tan of 30° and 60°.





a. What are the sizes of the angles θ, α and β?
b. What are the lengths of the sides marked h and a?
c. Write down the *exact* values of the lengths of the side h and a. (i.e. leave it as a square root)
d. Use the above triangles to fill in the trig ratios in the table below. Write the ratios as exact values.

   For example, cos 45°= $\frac{1}{\sqrt{2}}$. Use your calculator to check your answers, calculators usually give the answer as a decimal, for example, cos 45° ≈ 0.7071 which is 1 ÷ $\sqrt{2}$ rounded to four decimal places.

	0°	30°	45°	60°	90°
sine	0				1
cosine	1		$\frac{1}{\sqrt{2}}$		0
tangent	0				Does not exist

RADIAN MEASURE

Angles can be measured in degrees or radians. Radian measure is related to the arc length of a circle. One full turn of a circle is equivalent to 2π radians or 360°. Recall that π is a bit bigger than 3.

Rules for converting degrees and radians.

1. π radians is the same as 180°.
2. To convert radians to degrees, multiply by $180/\pi$
3. To convert degrees to radians, multiply by $\pi/180$
4. Keep in mind 1 radian is a slightly less than 60°.

Worked example

1. Convert 30° to radians.

   $180^\circ = \pi$ radians

   $1^\circ = \frac{\pi}{180}$ radians

   $1 \times 30^\circ = \frac{\pi}{180} \times 30$ radians

   $30^\circ = \frac{\pi}{6}$ radians

2. Convert $\frac{\pi}{12}$ radians to degrees.

   π radians$=180^\circ$

   $\frac{\pi}{12}$ radians$=\frac{180^\circ}{12}=15^\circ$

3. Convert 1.4 radians to degrees (to the nearest degree).

   π radians $= 180^\circ$

   1 radian $= \frac{180^\circ}{\pi}$

   1×1.4 radians $= \frac{180^\circ}{\pi} \times 1.4$

   1.4 radians $= \frac{252^\circ}{3.1415} = 80^\circ$

Exercise 9

1. Convert the following degrees to radians. Leave the answer in terms of π.

a. 45°

b. 60°

c. 90°

d. 225°

2. Convert the following radians to degrees. Give the answer correct to the nearest degree.

a. $\frac{3\pi}{8}$ radians

b. 3.1 radians

ANSWERS TO EXERCISES

SOME BASIC GEOMETRY

Exercise 1

1. a)II b)I c)V d)III e)IV

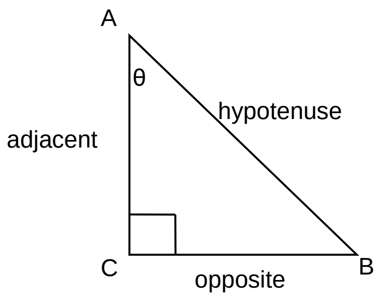
NAMING ANGLES

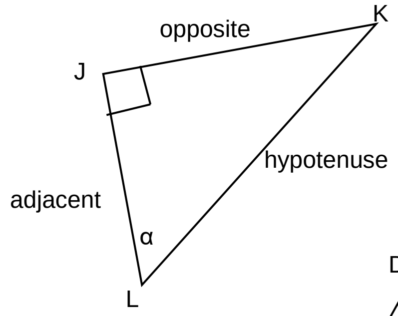
Exercise 2

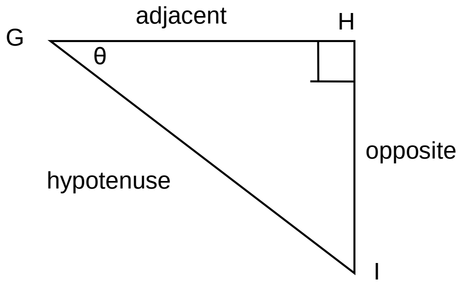
1. Obtuse<ABC, <FBE: Acute <ABE,<CBF
2. a=74° b=146° c=88° d=122° e=58° f=162° g=34° h=92°

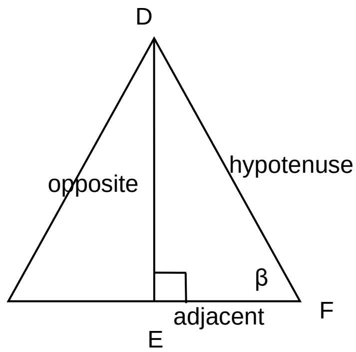
NAMING THE SIDES OF A RIGHT TRIANGLE

Exercise 3









SIMILAR TRIANGLES

Exercise 4

1. a. Scale factor is 2. b. Scale factor is 3.5.
2. $\frac{\text{tree height}}{2} = \frac{23}{5} = 4.6$ Tree Height = 4.6 x 2 = 9.2 metres.

TRIGONOMETRIC RATIOS

Exercise 5

Trig. Ratios	φ	α	θ	β
Sine	$\frac{8}{10}=\frac{4}{5}$	$\frac{5}{13}$	$\frac{8}{17}$	$\frac{2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}$
Cosine	$\frac{6}{10}=\frac{3}{5}$	$\frac{12}{13}$	$\frac{15}{17}$	$\frac{2}{4}=\frac{1}{2}$
Tangent	$\frac{8}{6}=\frac{4}{3}$	$\frac{5}{12}$	$\frac{8}{15}$	$\frac{2\sqrt{3}}{2}=\sqrt{3}$

FINDING AN UNKNOWN ANGLE

Exercise 6

1. $\gamma = \sin^{-1}$ (35/41) = 58.6° β = $\tan^{-1}$ (14.6/5.5) = 69.4°
2. $\tan\theta = \frac{9.2}{23} = \frac{2}{5} = 0.4$, $\therefore \theta = \tan^{-1}(0.4) = 21.80°$

FINDING AN UNKNOWN SIDE
Exercise 7

1. $\sin 52.9° = \frac{x}{41}$
   $x = 41\sin 52.9° = 32.7$

   $\tan 40° = \frac{14.6}{y}$

   $y = 14.6/\tan 40° = 17.4$ units

PYTHAGORUS THEOREM
Exercise 8

1. $5^2+12^2=25+144=169$, $\therefore h=\sqrt{169}=13$ Yes, right angle.

2. $2^2+1^2=4+1=5$, Length of diagonal = $\sqrt{5}$=224cm

3. $25^2-24^2=625-576=49$, Width of swimming pool $=\sqrt{49}=7$m

4. (a)$\theta=45^\circ, \alpha=60^\circ, \beta=30^\circ$ (b) $h^2=1^2+1^2=2$, so $h=\sqrt{2}\approx 1.4142$ and $a^2=2^2-1^2=3$, so $a=\sqrt{3}\approx 1.7321$ (c) $h=\sqrt{2}$, $a=\sqrt{3}$ (d)

	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Does not exist

RADIAN MEASURE
Exercise 9

1. (a)$45° = \frac{\pi}{180} \times 45$

   $\therefore 45° = \frac{\pi}{4}$ radians

   (b)$60° = \frac{\pi}{180} \times 60$

   $\therefore 60° = \frac{\pi}{3}$ radians

   (c)$90° = \frac{\pi}{180} \times 90$

   $\therefore 90° = \frac{\pi}{2}$ radians

   (d)$225° = \frac{\pi}{180} \times 225$

   $\therefore 225° = \frac{5\pi}{4}$ radians

2. (a) $\frac{3\pi}{8}$ radians$=\frac{3\times 180°}{8}=67.5°$ (b)3.1 radians$=\frac{180}{\pi}\times 3.1$ $\therefore$ 3.1 radians$\approx 178°$